SOME CHARACTERIZATIONS OF SLANT AND SPHERICAL HELICES IN MINKOWSKI 3-SPACE

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Abstract

In this paper, we are investigating that under which conditions of the geodesic curvature of unit speed curve γ that lies on S_1^2 or H_0^2, the curve α which is obtained by using γ, is a spherical helix or slant helix in Minkowski space.

Key Words: Geodesic curvature, Helices, Spherical Helices, Slant Helices, Sabban Frame, Minkowski space, Spacelike curve, Timelike curve.

1 Introduction

There are several studies in literature examining methodology to use spherical curves to construct some specialized curves. For example, Izuyama and Takeuchi [1], defined a way to construct Bertrand curves from the spherical curve whose spherical evolute coincides with the spherical Darboux image of the Bertrand curve. In addition to it, Encheva and Georgiev [2], have used a similar method and defined a way to construct Frenet curves from the spherical curve whose spherical evolute coincides with the spherical Darboux image of the Frenet curve.

By using the method explained in those paper [1] [2], it is realized before [3] that there are some characterizations to construct spherical helices and slant helices in Euclidean space. By the help this, in this paper, we are able to find necessary conditions to construct complete spherical helices and slant helices in Minkowski space R_1^3.

The contributions of this study is elaborated in the following way: In section 2 basic concepts of Minkowski 3-Space R_1^3 are explained and related some lemmas are additionally proved. In section 3, Spherical Helices on R_1^3 are disccuessed by indicating some examples. Similarly, in section 4, Slant Helices on R_1^3 are examined.

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2 Basic Concepts

Let us consider the Minkowski 3-Space R_1^3 with the Lorentzian inner product

$$\langle x,y\rangle = x_1y_1 + x_2y_2 - x_3y_3$$

where $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3) \in R^3$. The pseudo-norm of a vector x is given by $\|x\| = \sqrt{|\langle x,x\rangle|}$.

In the space R_1^3, the Lorentzian cross-product is defined as follows

$$x \wedge y = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2, \quad x_3y_1 - x_1y_3, \quad x_2y_1 - x_1y_2)$$

It's clearly seen that the cross-product has the following properties[4],

$$\begin{aligned}
&(i) \quad x \wedge y = -(y \wedge x) \\
&(ii) \quad \langle x \wedge y, z\rangle = det(x,y,z) \\
&(iii) \quad x \wedge (y \wedge z) = \langle x,y\rangle z - \langle x,z\rangle y \\
&(iv) \quad \langle x \wedge y, x \wedge y\rangle = (\langle x,y\rangle)^2 - \langle x,x\rangle \langle y,y\rangle \\
&(v) \quad \langle x \wedge y, x\rangle = 0 \quad , \quad \langle x \wedge y, y\rangle = 0
\end{aligned}$$

where $x,y,z \in R^3$.

A vector $v \in R_1^3$ is called spacelike if $\langle v,v\rangle > 0$ or $v = 0$, timelike if $\langle v,v\rangle < 0$, lightlike if $\langle v,v\rangle = 0$ and $v \neq 0$ [5].

Given a regular curve $\alpha(s) : I \subset R \to R_1^3$. We say that α is spacelike (resp. timelike, lightlike) at s if $\alpha'(s)$ is a spacelike (resp. timelike, lightlike) vector. The curve α is called spacelike (resp. timelike, lightlike) if it is for any $s \in I$ [5].

Now, we will define two surfaces in R_1^3,

$$H_0^2 = \left\{(x_1,x_2,x_3) \in R_1^3 : x_1^2 + x_2^2 - x_3^2 = -1\right\}$$

is called *Hyperbolic plane* [5].

$$S_1^2 = \left\{(x_1,x_2,x_3) \in R_1^3 : x_1^2 + x_2^2 - x_3^2 = 1\right\}$$

is called *De Sitter space* [5]. We will call them Hyperbolic sphere of radius 1 and Lorentzian sphere of radius 1 respectively.

For a unit speed non-lightlike curve α with a spacelike or timelike normal vector $N(s)$, the Frenet formulae are given in [5]. It's easy to calculate the formulae for arbitrary speed non-lightlike curves as follows.

If α is a timelike curve,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa\nu & 0 \\ \kappa\nu & 0 & \tau\nu \\ 0 & -\tau\nu & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1}$$

If α is a spacelike curve with a spacelike normal vector $N(s)$,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa\nu & 0 \\ -\kappa\nu & 0 & \tau\nu \\ 0 & \tau\nu & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2}$$

If α is a spacelike curve with a timelike normal vector $N(s)$,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa\nu & 0 \\ \kappa\nu & 0 & \tau\nu \\ 0 & \tau\nu & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{3}$$

where

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{det\,(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}, \nu = \sqrt{|\langle \alpha', \alpha' \rangle|}. \tag{4}$$

In the formulae above we denote unit tangent vector with $T(s)$, unit binormal vector with $B(s)$, unit normal vector with $N(s)$.

A regular timelike or spacelike curve α is a helix, if τ/κ is a constant function.

For a unit speed curve α in R^3_1, slant helix characterization is given in [6]. We can prove the three Lemmas below by using those characterizations with (1), (2), (3), and (4).

Lemma 1. *Let α be a timelike curve in R^3_1. Then α is a slant helix if and only if either one of the next two functions*

$$\frac{\kappa^2}{\nu\,(\tau^2 - \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \quad or \quad \frac{\kappa^2}{\nu\,(\kappa^2 - \tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \tag{5}$$

is constant everywhere $\tau^2 - \kappa^2$ does not vanish.

Lemma 2. *Let α be a spacelike curve in R^3_1 with a spacelike normal vector. Then α is a slant helix if and only if either one of the next two functions*

$$\frac{\kappa^2}{\nu\,(\tau^2 - \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \quad or \quad \frac{\kappa^2}{\nu\,(\kappa^2 - \tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \tag{6}$$

is constant everywhere $\tau^2 - \kappa^2$ does not vanish.

Lemma 3. *Let α be a spacelike curve in R^3_1 with a timelike normal vector. Then α is a slant helix if and only if the function*

$$\frac{\kappa^2}{\nu\,(\tau^2 + \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \tag{7}$$

is constant.

Some characterizations of Lorentzian spacelike unit speed spherical curves were investigated in [7] and [8]. We can prove the two Lemmas below by using those characterizations with (1), (2), (3), and (4).

Lemma 4. *Let α be a spacelike curve in R_1^3 with a spacelike normal vector. Image of α lies on a Lorentzian sphere (resp. Hyperbolic sphere) of radius r and center a if and only if*

$$\frac{1}{\kappa^2} - \left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)^2 = \pm r^2 (resp.) \tag{8}$$

where $r > 0 \in R, \kappa \neq 0, \tau \neq 0$.

Lemma 5. *Let α be a spacelike curve in R_1^3 with a timelike normal vector. Image of α lies on a Hyperbolic sphere of radius r and center a if and only if*

$$\frac{-1}{\kappa^2} + \left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)^2 = -r^2 \tag{9}$$

where $r > 0 \in R, \kappa \neq 0, \tau \neq 0$.

Now, we need to prove a new lemma to reach our goal.

Lemma 6. *Let α be a timelike curve in R_1^3. Image of α lies on a Lorentzian sphere of radius r and center a if and only if*

$$\frac{1}{\kappa^2} + \left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)^2 = r^2 \tag{10}$$

where $r > 0 \in R, \kappa \neq 0, \tau \neq 0$.

Proof. Let $\alpha(s)$ be a timelike curve in R_1^3. We know from the definitions;

$$\langle T, T\rangle = -1, \langle N, N\rangle = 1, \langle B, B\rangle = 1.$$

If α lies on a Lorentzian sphere with radius r and center a, we can write

$$\langle \alpha - a, \alpha - a\rangle = r^2$$

If we differentiate with respect to s and use (1), we will have

$$\langle \nu T, \alpha - a\rangle = 0.$$

Let's differentiate again then we will have

$$\langle N, \alpha - a\rangle = \frac{1}{\kappa}.$$

Differentiating one more time will give us,

$$\langle B, \alpha - a\rangle = \frac{-\kappa'}{\nu\tau\kappa}.$$

As we know we can write $\alpha(s) - a = \lambda_1 T + \lambda_2 N + \lambda_3 B$ then,

$$\begin{aligned}
\langle \alpha - a, \alpha - a \rangle &= r^2 \\
\left\langle \frac{1}{\kappa}N - \frac{\kappa'}{\nu\tau\kappa^2}B, \frac{1}{\kappa}N - \frac{\kappa'}{\nu\tau\kappa^2}B \right\rangle &= r^2 \\
\frac{1}{\kappa^2} + \left(\frac{\kappa'}{\nu\tau\kappa^2}\right)^2 &= r^2 \\
\frac{1}{\kappa^2} + \left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)^2 &= r^2
\end{aligned}$$

Conversely, if we show that the equation

$$a = \alpha + \frac{1}{\kappa}N - \frac{\kappa'}{\nu\tau\kappa^2}B$$

is constant, then a should be the center of the Lorentzian sphere. By differentiating with respect to s and using (10) we have

$$\begin{gathered}
a' = \alpha' - \left(\tfrac{1}{\kappa}\right)' N - \left(\tfrac{1}{\kappa}\right) N' - \left(\tfrac{1}{\nu\tau}\left(\tfrac{1}{\kappa}\right)'\right)' B - \tfrac{1}{\nu\tau}\left(\tfrac{1}{\kappa}\right)' B' \\
a' = \upsilon T - \left(\tfrac{1}{\kappa}\right)' N - \left(\tfrac{1}{\kappa}\right)(\nu\kappa T + \nu\tau B) - \left(\tfrac{1}{\nu\tau}\right)'\left(\tfrac{1}{\kappa}\right)' B - \tfrac{1}{\nu\tau}\left(\tfrac{1}{\kappa}\right)'' B + \tfrac{1}{\nu\tau}\left(\tfrac{1}{\kappa}\right)' \nu\tau N \\
a' = \upsilon T - \left(\tfrac{1}{\kappa}\right)' N - \left(\tfrac{1}{\kappa}\right)\nu\kappa T - \left(\tfrac{1}{\kappa}\right)\nu\tau B - \left(\tfrac{1}{\nu\tau}\right)'\left(\tfrac{1}{\kappa}\right)' B - \tfrac{1}{\nu\tau}\left(\tfrac{1}{\kappa}\right)'' B + \left(\tfrac{1}{\kappa}\right)' N \\
a' = -\left(\tfrac{1}{\kappa}\right)\nu\tau B - \left(\tfrac{1}{\nu\tau}\right)'\left(\tfrac{1}{\kappa}\right)' B - \tfrac{1}{\nu\tau}\left(\tfrac{1}{\kappa}\right)'' B \\
a' = -\left(\tfrac{1}{\kappa}\right)\nu\tau B - \left(\tfrac{1}{\nu\tau}\right)'\left(\tfrac{1}{\kappa}\right)' B - \tfrac{1}{\nu\tau}\left(\tfrac{1}{\kappa}\right)'' B
\end{gathered}$$

Now let's differentiate (10) with respect to s we have

$$\begin{gathered}
0 = 2\tfrac{1}{\kappa}\left(\tfrac{1}{\kappa}\right)' + 2\left(\tfrac{1}{\nu\tau}\right)\left(\tfrac{1}{\kappa}\right)'\left(\tfrac{1}{\nu\tau}\left(\tfrac{1}{\kappa}\right)'\right)' \\
0 = 2\tfrac{1}{\kappa}\left(\tfrac{1}{\kappa}\right)' - 2\left(\tfrac{1}{\nu\tau}\right)\left(\tfrac{1}{\kappa}\right)'\left[\left(\tfrac{1}{\nu\tau}\right)'\left(\tfrac{1}{\kappa}\right)' + \tfrac{1}{\nu\tau}\left(\tfrac{1}{\kappa}\right)''\right] \\
\tfrac{1}{\kappa} = \left(\tfrac{1}{\nu\tau}\right)\left[\left(\tfrac{1}{\nu\tau}\right)'\left(\tfrac{1}{\kappa}\right)' + \left(\tfrac{1}{\nu\tau}\right)\left(\tfrac{1}{\kappa}\right)''\right] \\
\tfrac{\nu\tau}{\kappa} = \left(\tfrac{1}{\nu\tau}\right)'\left(\tfrac{1}{\kappa}\right)' + \left(\tfrac{1}{\nu\tau}\right)\left(\tfrac{1}{\kappa}\right)''
\end{gathered}$$

Then $a' = 0$.

$\square$

Let γ be a non-lightlike unit speed spherical curve ($\langle \gamma(s), \gamma(s) \rangle = \pm 1, \left\| \gamma'(s) \right\| = 1$) with an arc-length parameter s and denote $\gamma'(s) = t(s)$ where $\gamma' = d\gamma/ds$. If we set a vector $p(s) = \gamma(s) \wedge t(s)$, by definition we have an orthonormal frame $\{\gamma, t, p\}$. This frame is called the Pseudo-Sabban frame of γ. So we will have the following Lemma [9], [10].

Lemma 7. *Let $\gamma(s)$ be a unit speed spherical curve in R^3_1 then,*

(i) If γ is a timelike curve on S_1^2 then,

$$\begin{aligned} \gamma^{'} &= t \\ t^{'} &= k_g p + \gamma \\ p^{'} &= k_g t \end{aligned} \tag{11}$$

(ii) If γ is a spacelike curve on S_1^2 then,

$$\begin{aligned} \gamma^{'} &= t \\ t^{'} &= -k_g p - \gamma \\ p^{'} &= -k_g t \end{aligned} \tag{12}$$

(iii) If γ is a spacelike curve on H_0^2 then,

$$\begin{aligned} \gamma^{'} &= t \\ t^{'} &= k_g p + \gamma \\ p^{'} &= -k_g t \end{aligned} \tag{13}$$

where $k_g = det\left(\gamma, t, t^{'}\right)$ the geodesic curvature of curve γ.

In [2] Encheva and Georgiev showed a way to construct all *Frenet curves* $(\kappa > 0)$ by the following formula

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a \tag{14}$$

where b is a constant number, a is a constant vector, γ is a unit speed curve on S^2 with the $Sabban\ frame$ above, and $k : I \to R$ is a function of class C^1. Moreover, they showed that the spherical curve γ is a circle if and only if the corresponding *Frenet curves* are cylindrical helices.

If we make the calculations we will see

$$\begin{aligned} \alpha^{'}(s) &= be^{\int k(s)ds}\gamma(s) \\ \alpha^{''}(s) &= be^{\int k(s)ds}\left(k(s)\gamma(s) + \gamma^{'}(s)\right) \\ \alpha^{'''}(s) &= be^{\int k(s)ds}\left(\left(k^2(s) + k^{'}(s)\right)\gamma(s) + 2k(s)\gamma^{'}(s) + \gamma^{''}(s)\right). \end{aligned} \tag{15}$$

If we calculate κ, τ, and ν of the curve α by using the equations at (4) and (15), we will find

$$\begin{aligned} \kappa(s) &= \tfrac{1}{be^{\int k(s)ds}} \\ \tau(s) &= \tfrac{k_g(s)}{be^{\int k(s)ds}} \ . \\ \nu(s) &= be^{\int k(s)ds} \end{aligned} \tag{16}$$

It's easy to see

$$\begin{aligned} \langle \alpha'(s), \alpha'(s) \rangle &= b^2 e^{2\int k(s)ds} \langle \gamma(s), \gamma(s) \rangle \\ T(s) &= \gamma(s) \\ T'(s) &= t(s) \end{aligned} \tag{17}$$

So, we can say if γ is a unit speed spacelike curve which lies on S_1^2 then α is a spacelike curve with a space like normal vector $N(s)$.

If γ is a unit speed spacelike curve which lies on H_0^2 then α is a timelike curve with a spacelike normal vector $N(s)$.

If γ is a unit speed timelike curve which lies on S_1^2 then α is a spacelike curve with a timelike normal vector $N(s)$.

3 Spherical Helices on Lorentzian spheres and Hyperbolic spheres

Firstly, we want to show, under which circumstances the equation (14) is a spherical helix on Lorentzian sphere.

Theorem 1. *If the curve γ is a unit speed spacelike curve with a constant geodesic curvature, which lies on S_1^2, the curve α defined by (14) is a spherical helix which lies on a Lorentzian sphere if and only if the function $k(s) = k_g tanh[(k_g)(s-b_1)]$ where $b_1 \in R$.*

Proof. From (15), (16), and (17), we know the curve

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$

is a spacelike curve with a space like normal vector $N(s)$. So we need to use (8). Let's take the derivate of (8) with respect to s. Then we will have,

$$\left(\frac{1}{\nu}\left[\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right]' - \frac{\tau}{\kappa}\right)(s) = 0$$

By putting (16) in this equation, we will have

$$\left(\frac{1}{be^{\int kds}}\left[\frac{1}{k_g}\left(be^{\int kds}\right)'\right]' - k_g\right)(s) = 0$$

$$\left(\frac{1}{k_g e^{\int kds}}\left[k' e^{\int kds} + k^2 e^{\int kds}\right] - k_g\right)(s) = 0$$

$$k'(s) + k^2(s) = {k_g}^2.$$

If we solve this differential equation, we will have

$$k(s) = k_g tanh[(k_g)(s-b_1)]$$

Conversely, if we take $k(s) = k_g tanh[(k_g)(s-b_1)]$ in (14) then

$$\int k(s)ds = \int k_g tanh[(k_g)(s-b_1)]ds.$$

Let $u = k_g(s-b_1) = k_g s - k_g b_1$ then $k_g ds = du$, by using these equations

$$\begin{aligned}\int k(s)\,ds &= \int \tanh u\,du \\ &= \ln\cosh u + \ln b_2 \\ &= \ln\left[b_2\cosh\left(k_g(s-b_1)\right)\right]\end{aligned}$$

we have

$$\begin{aligned}\alpha(s) &= b\int e^{\int k(s)ds}\gamma(s)ds + a \\ &= b\int e^{\int k_g tanh\left[\left(k_g\right)(s-b_1)\right]ds}\gamma(s)ds + a \\ &= b\int e^{\ln\left[b_2\cosh\left(k_g(s-b_1)\right)\right]}\gamma(s)ds + a \\ &= b\int b_2\cosh\left(k_g(s-b_1)\right)\gamma(s)ds + a.\end{aligned}$$

where $b_1, b_2 \in R$.

Now, we must show that curve α is spherical. If we use (8) to do it, we will have

$$\begin{aligned}r^2 &= \left(\left(\frac{1}{\kappa^2} - \left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)\right)^2\right)(s) \\ &= \left(b^2e^{2\int kds} - \left(\frac{1}{be^{\int kds}\frac{k_g}{be^{\int kds}}}\left(\frac{1}{\frac{1}{be^{\int kds}}}\right)'\right)^2\right)(s) \\ &= \left(b^2e^{2\int kds} - \left(\frac{1}{k_g}\left(be^{\int kds}\right)'\right)^2\right)(s) \\ &= \left(b^2e^{2\int kds} - \frac{b^2k^2}{k_g^2}e^{2\int kds}\right)(s) \\ &= \left(b^2e^{2\int kds}\left(1-\frac{k^2}{k_g^2}\right)\right)(s) \\ &= b^2{b_2}^2\cosh^2\left(k_g(s-b_1)\right)\left(1-\frac{\left(k_g tanh\left[\left(k_g\right)(s-b_1)\right]\right)^2}{k_g^2}\right) \\ &= b^2{b_2}^2\cosh^2\left(k_g(s-b_1)\right)\left(\frac{1}{\cosh^2\left(k_g(s-b_1)\right)}\right) \\ &= b^2{b_2}^2.\end{aligned}$$

Therefore, it can be said that the curve α lies on a Lorentzian sphere which has a radius $|bb_2|$. □

Now, we can give another theorem.

Theorem 2. *If the curve γ is a unit speed spacelike curve with a constant geodesic curvature, which lies on H_0^2, the curve α defined by (14) is a spherical helix which lies on Lorentzian sphere if and only if the function $k(s) = k_g tan[(k_g)(s-b_1)]$ where $b_1 \in R$.*

Proof. By using (10) instead of (8) in Theorem 1, the proof is similar. □

Theorem 3. *If the curve γ is a unit speed timelike curve with a constant geodesic curvature, which lies on S_1^2, the curve α defined by (14) is a spherical helix which lies on Hyperbolic sphere if and only if the function $k(s) = k_g tanh[(k_g)(s-b_1)]$ where $b_1 \in R$.*

Proof. By using (9) instead of (8) in Theorem 1, the proof is similar. □

Example 1. Let's take $\gamma(s) = \left\{cos(s), sin(s), \sqrt{2}\right\}$, we know that γ is a spacelike curve on H_0^2 with the geodesic curvature $\sqrt{2}$. Then due to Theorem 2,

$$k(s) = k_g tan[(k_g)(s-b_1)]$$

and

$$\alpha(s) = b\int b_2 \cos(k_g(s-b_1))\gamma(s)ds + a$$

where $b, b_1, b_2 \in R$. If we take $b = 2, b_1 = 0, b_2 = 1$ then we have

$$\begin{gathered}
\alpha_1(s) = -2\cos\left(\sqrt{2}s\right)\sin(s) + 2\sqrt{2}\cos(s)\sin\left(\sqrt{2}s\right) \\
\alpha_2(s) = 2\cos(s)\cos\left(\sqrt{2}s\right) + 2\sqrt{2}\sin(s)\sin\left(\sqrt{2}s\right) \\
\alpha_3(s) = 2\sin\left(\sqrt{2}s\right)
\end{gathered}$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ and $a = (0,0,0)$

Example 2. Let's take $\gamma(s) = \left\{\sqrt{2}\cos\left(s/\sqrt{2}\right), \sqrt{2}\sin\left(s/\sqrt{2}\right), 1\right\}$, we know that γ is a spacelike curve on S_1^2 with the geodesic curvature $\sqrt{2}$. Then due to Theorem 1,

$$k(s) = k_g tanh[(k_g)(s-b_1)]$$

and

$$\alpha(s) = b\int b_2 \cosh(k_g(s-b_1))\gamma(s)ds + a$$

where $b, b_1, b_2 \in R$. If we take $b = 2, b_1 = 0, b_2 = 1$ then we have

$$\begin{gathered}
\alpha_1(s) = 2\cosh\left(s/\sqrt{2}\right)\sin\left(s/\sqrt{2}\right) + 2\cos\left(s/\sqrt{2}\right)\sinh\left(s/\sqrt{2}\right) \\
\alpha_2(s) = -2\cos\left(s/\sqrt{2}\right)\cosh\left(s/\sqrt{2}\right) - 2\sin\left(s/\sqrt{2}\right)\sinh\left(s/\sqrt{2}\right) \\
\alpha_3(s) = 2\sqrt{2}\sinh\left(s/\sqrt{2}\right)
\end{gathered}$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ and $a = (0,0,0)$

Example 3. Let's take $\gamma(s) = \left\{ \frac{1}{\sqrt{3}} \cosh\left(\sqrt{3}s\right), \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}} \sinh\left(\sqrt{3}s\right) \right\}$, we know that γ is a timelike curve on S_1^2 with the geodesic curvature $\sqrt{2}$. Then due to Theorem 3,

$$k(s) = k_g tanh\left[(k_g)(s - b_1)\right]$$

and

$$\alpha_1(s) = -2\sqrt{\tfrac{2}{3}} \cosh\left(\sqrt{3}s\right) \sinh\left(\sqrt{2}s\right) + 2\cosh\left(\sqrt{2}s\right) \sinh\left(\sqrt{3}s\right)$$
$$\alpha_2(s) = \frac{2\sinh\left(\sqrt{2}s\right)}{\sqrt{3}}$$
$$\alpha_3(s) = 2\cosh\left(\sqrt{2}s\right) \cosh\left(\sqrt{3}s\right) - 2\sqrt{\tfrac{2}{3}} \sinh\left(\sqrt{2}s\right) \sinh\left(\sqrt{3}s\right)$$

where $b, b_1, b_2 \in R$. If we take $b = 2, b_1 = 0, b_2 = 1$ then we have

$$\alpha(s) = \{2\sinh(s), 0, 2\cosh(s)\}$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ and $a = (0,0,0)$

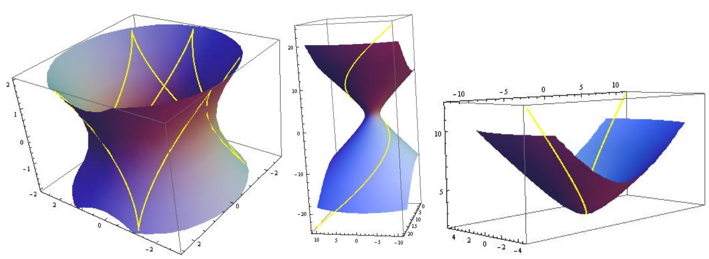

Figure 1: Spherical Helices (Resp. Example 1,2, and 3)

4 Slant Helices on Lorentzian spheres and Hyperbolic spheres

In this section, we want to give some characterizations about slant helices.

Theorem 4. *Let $\gamma(s)$ be a unit speed spacelike curve on S_1^2 ; b, m, n be constant numbers; and a be a constant vector. The geodesic curvature of $\gamma(s)$ satisfies*

$$k_g{}^2(s) = \frac{(ms+n)^2}{1+(ms+n)^2}$$

if and only if

$$\alpha(s) = b\int e^{\int k(s)ds} \gamma(s)ds + a$$

is a spacelike slant helix with a spacelike normal vector.

Proof. Let, for γ

$$k_g{}^2(s) = \frac{(ms+n)^2}{1+(ms+n)^2}. \tag{18}$$

From (15), (16), and (17), we know $\alpha(s)$ is a spacelike curve with a space like normal vector $N(s)$. So, from (6) the geodesic curvature of the spherical image of the principal normal indicatrix of α is as follows

$$\begin{aligned}\sigma(s) &= \left(\frac{\kappa^2}{\nu\left(\kappa^2-\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s)\\ &= \left(\frac{\frac{1}{\nu^2}}{\nu\left(\frac{1}{\nu^2}-\frac{k_g{}^2}{\nu^2}\right)^{3/2}}k_g{}'\right)(s).\end{aligned}$$

So we have

$$\sigma(s) = \frac{k_g{}'(s)}{\left(1-k_g{}^2(s)\right)^{3/2}} \tag{19}$$

Now, let's take $u(s) = ms+n$ then we have (18)

$$k_g{}^2(s) = \frac{u^2(s)}{1+u^2(s)}. \tag{20}$$

If we take the derivates of the both sides of (20) with respect to s we have

$$\begin{aligned}2k_g(s)\,k_g{}'(s) &= \left(\frac{2uu'\left(1+u^2\right)-\left(2uu'\right)u^2}{\left(1+u^2\right)^2}\right)(s)\\ k_g(s)\,k_g{}'(s) &= \left(\frac{uu'}{\left(1+u^2\right)^2}\right)(s)\\ k_g{}'(s) &= \left(\left(\frac{uu'}{\left(1+u^2\right)^2}\right)\left(\varepsilon\sqrt{\frac{1+u^2}{u^2}}\right)\right)(s) \quad (21)\end{aligned}$$

where $\varepsilon = \pm 1$. Putting (20) and (21) in (20), we have

$$\begin{aligned}
\sigma(s) &= \frac{k_g{}'(s)}{\left(1 - k_g{}^2(s)\right)^{3/2}} \\
&= \left(\varepsilon \frac{\frac{\sqrt{1+u^2}uu'}{|u|\left(1+u^2\right)^2}}{\left(\frac{u^2}{1+u^2}+1\right)^{3/2}} \right)(s) \\
&= \left(\varepsilon \frac{\sqrt{1+u^2}uu'}{|u|\left(1+u^2\right)^2} \left(1+u^2\right)^{3/2} \right)(s) \\
&= \left(\varepsilon \frac{\left(1+u^2\right)^2}{\left(1+u^2\right)^2} \frac{u}{|u|} u' \right)(s) \\
&= \varepsilon \frac{ms+n}{|ms+n|} m \\
&= \varepsilon m
\end{aligned}$$

which is constant.

Conversely, let $\alpha(s)$ be a spacelike slant helix, then the geodesic curvature of the spherical image of the principal normal indicatrix of α is a constant function. So we can take

$$\sigma(s) = \left(\frac{\kappa^2}{\nu\left(\kappa^2 - \tau^2\right)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \right)(s) = m$$

where $m \in R$. Therefore, from (19)

$$\begin{aligned}
m &= \left(\frac{\kappa^2}{\nu\left(\kappa^2 - \tau^2\right)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \right)(s) \\
&= \frac{k_g{}'(s)}{\left(1 - k_g{}^2(s)\right)^{3/2}}
\end{aligned}$$

If we solve this differential equation, we have

$$\frac{k_g(s)}{\sqrt{1 - k_g{}^2(s)}} = ms + n$$

where $n \in R$. Then,

$$\frac{k_g{}^2(s)}{1-k_g{}^2(s)} = (ms+n)^2$$

$$\frac{k_g{}^2(s)+1-1}{1-k_g{}^2(s)} = (ms+n)^2$$

$$\frac{1}{1-k_g{}^2(s)} - 1 = (ms+n)^2$$

$$k_g{}^2(s) = 1 - \frac{1}{1+(ms+n)^2}$$

$$k_g{}^2(s) = \frac{(ms+n)^2}{1+(ms+n)^2}.$$

□

Theorem 5. *Let $\gamma(s)$ be a unit speed spacelike curve on H_0^2; b, m, n be constant numbers; and a be a constant vector. The geodesic curvature of $\gamma(s)$ satisfies*

$$k_g{}^2(s) = \frac{(ms+n)^2}{1+(ms+n)^2}$$

if and only if

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$

is a timelike slant helix with a spacelike normal vector.

Proof. By using (5) instead of (6) in Theorem 4, the proof is similar. □

Theorem 6. *Let $\gamma(s)$ be a unit speed timelike curve on S_1^2; b, m, n be constant numbers; and a be a constant vector. The geodesic curvature of $\gamma(s)$ satisfies*

$$k_g{}^2(s) = \frac{(ms+n)^2}{1-(ms+n)^2}$$

if and only if

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$

is a spacelike slant helix with a timelike normal vector.

Proof. By using (7) instead of (6) in Theorem 4, the proof is similar. □

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